Primes in an integral extension

If $R \subseteq S$ and S is integral over R, we want to know how to compare $\operatorname{Spec} R$ to $\operatorname{Spec} S$.

Thm (Lying over + going up): Suppose $R \subseteq S$ is an integral extension. Let $P \subseteq R$ be a prime ideal. There exists a prime Q of S w/ $R \cap Q = P$. In fact Q can be chosen to contain any ideal Q_0 that satisfies $R \cap Q_0 \subseteq P$.

(First part is "Lying over", second is "Going up.")

Note that this implies that the corresponding map on Spec is surjective! However, we do need integrality for this:

Ex: Define $\varphi : k[t] \to k[x,y]/(xy-1)$,

$$t \longmapsto x$$

This is injective! However, the corresponding map on Spec is not surjective: The nonzero primes in $k[x,y]$ are $P_{a,b} = (x-a,\, y-b)$ s.t. $ab = 1$. In particular, $\varphi^{-1}(P_{a,b}) \neq (t)$ for any a, b.

Pf of Thm: Modding out by Q_0 and $R \cap Q_0$, we can assume $Q_0 = 0$. Thus, we just need to show $\exists$ a prime Q of S s.t. $R \cap Q = P$.

Let $U = R - P$. Replacing R by R_P and S by $S[U^{-1}]$, assume R is local w/ max'l ideal P.

Any max'l ideal of S containing PS has preimage containing P, and therefore equal to P. Thus, we just need to show $PS \neq S$.

If $PS = S$, then 1 is an S-linear comb. of elements of P. Let S' be the subalgebra generated by these coefficients.

Then $1 \in PS'$, so $PS' = S'$. Since S' is integral $/R$, S' is a f.g. R-module. By Nakayama, $S' = 0$, which is a contradiction. $\square$

From this, we obtain the classical "Going up" Theorem, which says

If $P_0 \subseteq P_1 \subseteq \dots \subseteq P_d$ is a chain of prime ideals in R, $\exists$

$Q_0 \subseteq Q_1 \subseteq \dots \subseteq Q_d$ a chain of prime ideals in S

such that Q_i "lies over" P_i. i.e. $Q_i \cap R = P_i$. To construct the Q_i's we first find Q_0. Then from the theorem, we can find $Q_1 \supseteq Q_0$ st Q_1 lies over P_1. The rest follows inductively.

There is also a "going down theorem" that holds under stronger hypotheses. (see ch. 13).

We get a similar result even if we assume the field of fractions of the big ring is algebraic over that of the smaller:

Lemma: Let $R \subsetneq S$ be integral domains. If $K(S)$ is algebraic over $K(R)$, then any nonzero ideal of S intersects R nontrivially.

Pf: Notice it suffices to prove the lemma for a principal ideal $(b) \in S$. then b satisfies some eqn $a_n b^n + \dots + a_1 b + a_0 = 0$ w/ $a_i \in K(R)$, $a_0 \neq 0$. Then we can bultiply through by a common denominator in R, and assume all $a_i \in R$.

Thus $\underset{\neq 0}{a_0} \in (b) \cap R$. □

If we assume in addition that R is a field, then any nonzero ideal of S contains <u>all</u> of R, so in particular it contains 1. Thus, S is a field. In fact we have the following.

Cor: If $R \subseteq S$ are domains, and S integral over R, then S is a field iff R is a field.

Pf: We already showed that R a field $\Longrightarrow$ S is too.

If S is a field, take $m \subseteq R$ a max'l ideal. Then $\exists$ a prime $Q \subsetneq S$ s.t. $Q \cap R = m$. But $Q = 0$, so $m = 0$, so R is a field. □

Note that if $P \subseteq S$ is an ideal, then $R/R\cap P \subseteq S/P$ is again an integral extension, so the corollary implies P is max'l $\Longleftrightarrow$ $R \cap P$ max'l.

We can also deduce that two primes that have the same intersection in R must be incomparable:

Cor: If $R \subseteq S$ is an integral extension, then if $Q \subseteq Q_1 \subseteq S$ are primes s.t. $Q \cap R = Q_1 \cap R$, then $Q = Q_1$.

Pf: Let $P = Q \cap R$. Then S/Q is integral over R/P. But then $K(S/Q)$ is algebraic over $K(R/P)$, since S/Q is integral over $K(R/P)$, and thus its algebraic closure must be $K(S/Q)$.

So by the lemma, $Q_1/Q = 0 \Longrightarrow Q = Q$. $\square$